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UNIVERSITY OF NAMIBIA

Faculty	Agriculture, Engineering and Natural Sciences
School	Science
Department	Computing, Mathematical and Statistical Science
Subject	Calculus I (Old Curriculum)
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Marks	100

FIRST OPPORTUNITY EXAMINATION PAPER

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INSTRUCTIONS:

(i) This question paper consists of **FIVE** pages (***including*** this front page).

(ii) Answer **ALL** questions in section A and **ANY** 3 out of 4 questions in section B.

(iii) Only ***non-programmable calculators*** may be used.

(iv) Try to understand each question before you answer it.

(v) Number the questions clearly and present your solutions in a logical manner.

(vi) Use proper mathematical terminology.

(vii) The full marks for this paper is 100.

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EXAMINATIONS

Section A.[40 marks]

Answer **ALL** questions in this section.

Question A1.[6 marks]

Give a precise definition of the following concepts.

a) $c \in \mathbb{R}$ is an accumulation point of a subset A of $\mathbb{R}$. [2]

b) $\lim_{x \to c^+} f(x) = -\infty$, where $c \in \mathbb{R}$ is an accumulation point of D_f. [4]

Question A2.[8 marks]

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

Let $I \subseteq \mathbb{R}$ be an interval with $c \in I$ and $f : I \to \mathbb{R}$ a function.

a) If f is differentiable at c, then both one sided derivatives of f exist at c. [4]

b) If $\lim_{x \to c} f(x) = L \in \mathbb{R}$, then both one sided limits of f exists at c and they are both equals to L. [4]

Question A3.[18 marks]

a) Verify the following statements by applying the definition.

(i) $\lim_{x \to 2} \frac{x-1}{x^2} = \frac{1}{4}$. [6]

(ii) Let $a, b \in \mathbb{R}$. The function $f(x) = \sqrt{x-a} + b$ is continuous at a. [4]

b) Solve the following equation/inequality for $x \in \mathbb{R}$. In the following

$$\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\} \text{ is the greatest integer/floor function}$$

and

$$\lceil x \rceil := \min\{z \in \mathbb{Z} : z \geq x\} \text{ is the least integer/ceiling function.}$$

(i) $\lceil \lfloor x \rfloor \rceil = 3$. [2]

(ii) $4 \leq \lceil x^2 + 2x \rceil \leq 15$. [6]

Question A4.[8 marks]

a) Use the Squeeze theorem to show that

$$\lim_{x \to -\infty} \left(\frac{\lceil x \rceil}{x + 2023} \right) = 1,$$

where $\lceil \cdot \rceil$ is the least integer/ ceiling function defined as in question A3 above. Hint: $u - 1 < \lceil u \rceil < u + 1$ for all $u \in \mathbb{R}$. [4]

b) Evaluate $\displaystyle\lim_{x \to \sqrt{\pi}} \left(\frac{ae^{\sin(x^2)} - a}{\pi - x^2} \right)$, where $a \in \mathbb{R} - \{0\}$. [4]

Section B [60 marks]

Answer **ANY** 3 OUT OF 4 questions in this section.

Question B1. [20 marks]

a) By first computing each of the one sided limit, compute the limit if it exists and explain why, if it does not exist.

$$\lim_{x \to 1} \left(\frac{\lceil x^2 + 1 \rceil}{\lfloor x^2 + 1 \rfloor} \right)$$

, where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the least integer/ceiling function and greatest integer/floor function defined as in question A3. [7]

b) Consider the function $f(x) = \cos\left(\frac{1}{x}\right)$.

(i) Give the domain of f and show that 0 is an accumulation point of the domain of f. [3]

(ii) Come up with two sequences $(x_n)_{\mathbb{N}}$ and $(y_n)_{\mathbb{N}}$ in $\mathbb{R}$ with

1. $\lim_{n \to \infty} x_n = 0$, [2]

2. $\lim_{n \to \infty} y_n = 0$, [2]

3. $\lim_{n \to \infty} \cos\left(\frac{1}{x_n}\right) = 0$, [2]

4. $\lim_{n \to \infty} \cos\left(\frac{1}{y_n}\right) = -1$, [2]

5. With reasons, conclude whether $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ exists or not. [2]

Question B2. [20 marks]

Consider the function

$$f(x) = \ln\left|\frac{1 - xe^2}{x+1}\right|$$

a) Find the domain D_f of f. [1]

b) Find the x and y-intercepts. [2]

c) Find $\lim_{x\to c} f(x)$, where c is an accumulation point of D_f which is not in D_f . Identify any possible asymptotes. [3]

d) Find $\lim_{x\to\pm\infty} f(x)$. Identify any possible asymptote. [2]

e) Find $f'(x)$ and $f''(x)$. [3]

f) Find the intervals of increase and decrease. [3]

g) Discuss the concavity of f and give any possible point(s) of inflection. [3]

h) Sketch a well labeled graph of f. [3]

Question B3. [20 marks]

a) Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$ for all $x \in \mathbb{R}$. [5]

b) Find the derivative y', where $y = \sec^{-1}\left(2^x + \sinh(2^x)\right)$. [5]

c) Show that of all the rectangles with a given area A, the one with smallest perimeter is a square. [3]

d) A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m? [7]

Question B4. [20 marks]

a) Express $25\cosh x - 24\sinh x$ in the form $R\cosh(x-\alpha)$ giving the values of R and $\tanh\alpha$, where $R, \alpha \in \mathbb{R}$. [5]

b) If $f(x) = 25\cosh x - 24\sinh x$, use your answer in a) above to find the critical number of f and classify it. [4]

c) If f is continuous on $\mathbb{R}$ and $a \in \mathbb{R}$, prove that

$$\int_a^{\sqrt{1+a^2}} x\sqrt{4+4a^2}\, f(x^2-a^2)dx = \sqrt{1+a^2}\int_0^1 f(x)\ dx.$$

[4]

d) Evaluate the following integrals.

(i) $\displaystyle\int_{\pi/2}^{3\pi/4} \sin^5(2x)\cos^4(2x)\ dx.$ [4]

(ii) $\displaystyle\int_0^{\pi/2} \cos(x)\sin(10x)\ dx.$ [3]

END